J. Fluid Mech. (1990), *vol.* 213, *pp.* 349–375
Printed in Great Britain

Plane Stokes flow driven by capillarity on a free surface

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(Received 26 May 1989)

The free creeping viscous incompressible plane flow of a finite region, bounded by a simple smooth closed curve and driven solely by surface tension, is analysed. The shape evolution is described in terms of a time-dependent mapping function $z = \Omega(\zeta, t)$ of the unit circle, conformal on $|\zeta| \leqslant 1$. An equation giving the time evolution of the $\Omega(\zeta, t)$ is derived. In practice, it has been necessary to guess a parametric form, i.e. $\Omega(\zeta, t) = \Omega[\zeta; a_1(t), a_2(t), \ldots]$, whose validity must be verified using the shape-evolution equation. Polynomial and proper rational mappings with no repeated factors are apparently always valid in principle. Solutions are given for (i) regions bounded initially by a regular epitrochoid, (ii) the limiting case of a half-plane bounded by a trochoid, and (iii) a class of rosettes whose mapping is rational. The two-lobed rosette gives the exact solution of the coalescence of equal cylinders. All these mappings involve limiting initial shapes having inward-pointing cusps. Useful parameterizations providing regions whose limiting shapes possess corners or outward-pointing cusps have not been found.

1. Introduction

This article addresses a special type of moving free-boundary problem in fluid dynamics: briefly, creeping viscous incompressible plane flow in a finite region, bounded by a simple smooth closed curve and driven solely by surface tension. Such problems are self-contained in that the applied tractions are intrinsic in the geometry. The objective is to determine exactly the time evolution of the shape of the region. The problems are fundamentally nonlinear owing to the large changes in shape, and it is emphasized that no mathematical approximations are made.

The intent of the article being the elucidation of a physical process, the mathematics itself has accordingly been kept informal. The author believes that sufficient conditions for the rigorous justification of the conclusions are that the boundary initially be as stated in the opening paragraph – simple, smooth and closed. The smoothness of the initial boundary will be adequate if the curvature is a differentiable function of the distance along the curve (cf. footnote on p. 000). The boundary curve is constrained in this way also for physical reasons, but mathematically it may approach a boundary with corners or cusps arbitrarily closely.

The time-dependent shape of the region in the complex z-plane is described in terms of a time-dependent conformal mapping function $\Omega(\zeta, t)$ on the fixed region $|\zeta| \leqslant 1$ of the complex ζ-plane. Such as description is always possible (Henrici 1974 §5.10, 1986 §16.3). An equation giving the time evolution of $\Omega(\zeta, t)$ is derived. In practice, it has been necessary to conjecture a parametric form, i.e. $\Omega(\zeta, t) = \Omega[\zeta;$

$a_1(t), a_2(t), \ldots]$, whose validity must be verified using the shape-evolution equation. Suitable parameterizations are not always obvious. It is argued that certain forms (e.g. polynomials) are always valid, but solving the equations themselves (i.e. for the $a_n(t)$) can be difficult. When the conjectured form holds and the equations can be solved, the evolution of the shape with time is obtained in simple, exact and closed form. Internal velocity and stress fields can then be obtained. Some cases involving finite regions provide limit cases involving infinite ones. A separate article will deal with other unbounded regions (Hopper 1990*a*).

Except for linearized small-amplitude problems and wave motion, there are relatively few exact solutions to fluid-dynamical problems involving moving free boundaries. Keller & Miksis (1983) give a similarity solution of a free moving-boundary problem. The notion of describing the shape evolution of a free surface in terms of a time-dependent conformal mapping is so obvious that it surely must have occurred to people long ago. Time-independent Stokes flows have been described in terms of conformal maps by, e.g. Garabedian (1966) and by Richardson (1968, 1973), and time-independent potential flow may be treated using the well-known hodograph method (e.g. Carrier, Krook & Pearson, 1966 §4–6). The earliest work on time-dependent flows of which the author is aware is that of Galin (1945) on seepage in porous beds. The work of Richardson (1972) on Hele-Shaw flows proved seminal: see also Richardson (1982), Elliott & Ockendon (1982), Howison (1986*a*, 1986*b*). It may be noted that Richardson provided a constructive method, while the present article does not. Seepage and Hele-Shaw flows are potential flows satisfying Laplace's equation (Batchelor 1967). Conformal maps have also been used to describe time-dependent shapes in other fields (e.g. Shriaman & Bensimon 1984). The present work addresses time-dependent free-boundary Stokes flow. The study arose from the author's interest in sintering, which led him to solve exactly the problem of the viscous coalescence of two equal cylinders (Hopper 1984). The theory is a generalization of concepts implicit in that work, resulting in much easier methods.

The successful application of the theory requires the discovery – perhaps based on intuition – of a form of $\Omega(\zeta, t)$ that is suitable to the problem at hand. It may be helpful to begin with a specific example: the nephroid (two-cusped, two-lobed epicycloid) is described by the mapping

$$z = \omega(\zeta) = (\zeta - \tfrac{1}{3}\zeta^3) \quad (|\zeta| = 1). \tag{1}$$

Consider the z-plane image of a smaller ζ-plane circle, $|\zeta| = r < 1$. This resembles the nephroid except that the cusps have become rounded and the area enclosed, smaller. The images of yet smaller ζ-circles become rounder and smaller still (they are epitrochoids); and as the ζ-circles approach zero radius, their z-images approach tiny circles. Qualitatively, then, the sequence of images described resembles what would intuitively be expected of creeping flow driven by surface tension, except that the area should not change. As might be supposed, it is convenient to deal with an unchanging region in the ζ-plane. So a mapping function is constructed in terms of (1) in such a way that the sequence of z-images are controlled by a time-dependent parameter $\lambda(t)$ representing the decreasing ζ-radii, and such that the changing area is avoided. The following convenient form meets this prescription:

$$z = \Omega(\zeta, t) = B[\lambda(t)]\, \omega[\lambda(t)\zeta] \quad (|\zeta| \leqslant 1), \tag{2}$$

$$B(\lambda) = \lambda^{-1}(1 + \tfrac{1}{3}\lambda^4)^{-\frac{1}{2}}, \tag{3}$$

$B(\lambda)$ has been chosen to give an enclosed area of π. The mapping is obviously

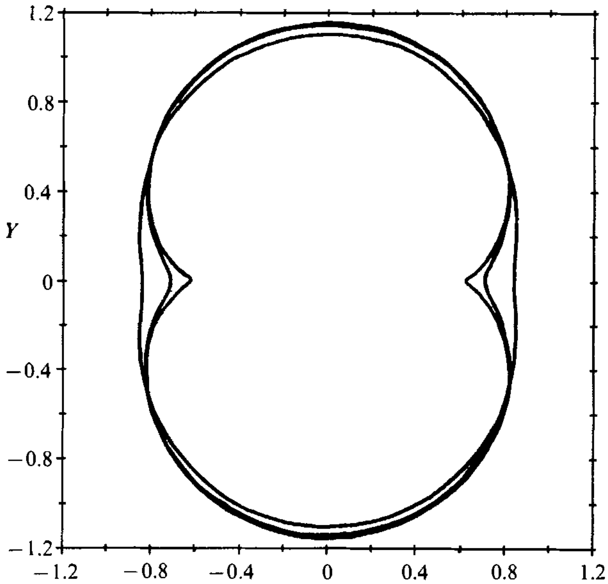

FIGURE 1. The nephroid family – the $N = 2$ epitrochoids of area π (§4.1), $\nu = \lambda^2 < 1$: $\nu = 0.9$ ($t \approx 0.0295$), $\nu = 0.7$ ($t \approx 0.147$), $\nu = 0.4$ ($t \approx 0.551$). The cusps of the nephroid ($\nu = 1$) occur at $z = \pm 0.577$.

conformal on $|\zeta| \leqslant 1$ when $\lambda < 1$. This parameterization is natural for cases involving a high degree of symmetry, and we shall frequently use – or attempt to use – it in subsequent sections. Equations (1)–(3) now describe a sequence of one-parameter curves (figure 1) that qualitatively describe the subsequent shape evolution of any one. That is, given a region bounded initially by the curve corresponding to $\lambda_i < 1$, its subsequent shape evolution by creeping plane incompressible flow driven by the surface tension of its boundary resembles qualitatively the continuous sequence of curves parameterized by decreasing λ. As proven below, this description is in this case (and in many others) more than qualitative; *it is exactly true for a unique function* $\lambda(t)$. This solves the flow problem. For both mathematical and physical reasons, the singular initial condition $\lambda_i = 1$ is not included, and we impose the restriction $\lambda_i < 1$; mathematically, however, λ_i may be arbitrarily close to one. It is natural to choose the time zero such that $\lambda(0) = 1$; then the solution applies to $t > 0$.

The fact that certain shapes evolve very simply, when viewed as conformal maps, is not only beautiful but of fundamental interest. That the two stress and one kinematic boundary conditions involved can simultaneously be satisfied in this way seems almost miraculous, but it is also suggestive of deep connections between geometry, mappings and dynamics in this class of problems. If so, insights into those connections may lead to methods applicable to less restricted flows. More immediately, the analytical tools developed have direct application to various scientific issues, including the healing of cracks in glass and sintering, matters to be addressed elsewhere (Hopper 1990*b*).

The key feature giving this description its utility is that the essence of the mapping function does not change with time. Thus, the map remains conformal with two critical points. Though the critical points ($\zeta = \pm 1/\lambda$) move with time, their motion does not change the symmetry, they do not change their nature [simple zeros of $\Omega'(\zeta)$], they do not disappear off the ζ-plane, nor do new ones appear. These statements would also apply to a distorted nephroid in which two critical points are located asymmetrically (i.e. at $\zeta = +1/\lambda, -1/\mu$), but determining $\lambda(t)$ and $\mu(t)$ proves difficult (§4.3). No such simple situation exists even in principle when the critical points are branch points (arising for such interesting cases as the interior of a polygon): applying the notion of moving but otherwise unchanging critical points to branch points gives an intuitively satisfying qualitative behaviour, but it is not quantitatively correct.

2. Formulation and the shape-evolution equation

A criterion for the truth of a chosen shape-evolution conjecture will now be derived. Zero-subscripted variables will denote actual dimensional ones, and unsubscripted variables will be reduced dimensionless ones. Let Γ_0 be a simple, smooth closed curve in the complex z_0-plane. Denote by D_0 the open (finite, simply-connected) region bounded by Γ_0. Choose a characteristic length R_0, for example such that the area of D_0 is πR_0^2. We shall regard D_0 as the cross-section of an infinitely long isothermal general cylinder of Newtonian viscous liquid having dynamic viscosity η, density ρ and surface tension γ, in a gravitational field $\boldsymbol{g}$, all these being constants. Consider plane flow: that is, the velocity vector is independent of and normal to the axial coordinate, and this is true also of the other vector quantities. Using vector notation for now, denote the position $\boldsymbol{x}_0$, velocity $\boldsymbol{u}_0$, curvature κ_0, force vector per unit length (of the cylinder) $\boldsymbol{F}_0$, stress tensor components $\tau_{0,ij}$, pressure p_0, and time t_0. The surface tractions driving the flow arise from the surface tension, and the reduction to dimensionless form is chosen accordingly. Therefore, define the dimensionless variables $\boldsymbol{x} = \boldsymbol{x}_0/R_0$, $\kappa = \kappa_0 R_0$, $\boldsymbol{F} = \boldsymbol{F}_0/\gamma$, $\tau_{ij} = \tau_{0,ij}R_0/\gamma$, $p = p_0 R_0/\gamma$, $\boldsymbol{u} = \eta \boldsymbol{u}_0/\gamma$, $t = \gamma t_0/\eta R_0$. Then the Navier–Stokes momentum equation in dimensionless form becomes

$$\frac{\rho\gamma R_0}{\eta^2}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla^2\boldsymbol{u} - \boldsymbol{\nabla} p + \frac{\rho R_0^2 \boldsymbol{g}}{\gamma}. \tag{4}$$

When the two dimensionless groups (the Suratman and Bond numbers; see Bolz & Tuve 1973) are both small, the inertial and gravitational forces are small in comparison with both viscous and capillary forces, and (4) reduces to Stokes equations (momentum and continuity) in dimensionless form:

$$\nabla^2\boldsymbol{u} = \boldsymbol{\nabla} p, \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{5}$$

The stress boundary condition in the formalism below is that the surface traction is in the outward normal direction and of magnitude κ. The circumstances under which this model is a good physical approximation are discussed in §5. As our interest is in the shape evolution, rigid-body motions are ignored. (It is interesting that the dimensional velocity $\boldsymbol{u}_0$ is independent of the size of the region; of course, it therefore changes a small region faster than a large.)

It is convenient to exploit the equivalence of Stokes equations to the equilibrium and compatibility equations of elasticity for an incompressible material. (In dimensional form, the velocity is replaced by the displacement and the viscosity by the shear modulus, and Poisson's ratio is $\frac{1}{2}$.) The plane creeping viscous-flow problem is then equivalent to an incompressible plane-strain elasticity problem. The latter may be treated using a stress function, which satisfies the biharmonic equation. Any biharmonic function of two real variables (x, y) can be expressed in terms of two analytic functions of the complex variable $z = x + \mathrm{i}y$ using the Goursat representation. When this is done for the stress function, one obtains the Kolosoff–Muskhelishvili equations relating two analytic functions $\phi(z)$ and $\psi(z)$ to the surface tractions, displacements and stresses. The historical development of all this is referenced by Muskhelishvili (1953*a*).

An equivalent formulation would begin by applying the Goursat representation to a Lagrange stream function. This results in equations to (7) and (8) below, the differences being essentially notational (Richardson 1968). The elasticity formulation was preferred for being more readily available in well-organized textbook form – for example, in Muskhelishvili's classic monograph (1953*a*) and in Sokolnikov's (1956) briefer version.

In the following, primes and parenthentical exponents denote complex derivatives (independent of the direction of approach) with respect to the independent (complex) variable; a dot denotes the derivative with respect to time; and an asterisk denotes the complex conjugate. Also, in the ζ-plane let $\sigma = e^{i\vartheta}$. Thus, ζ will be used to denote general points in the ζ-plane, while σ will always indicate points on the unit circle of the ζ-plane. Note that $\sigma^* = 1/\sigma$. By $d/d\sigma$ is meant the derivative along the curve $|\sigma| = 1$; i.e. $d\sigma = i\, e^{i\vartheta}\, d\vartheta$.

The derivation of the shape-evolution condition proceeds along the following lines: a kinematic condition is obtained relating $\dot{\Omega}(\sigma, t)$ to the surface velocities. This determines $\phi(\zeta)$ in terms of a Cauchy integral involving $\Omega'(\sigma, t)$. Simultaneously requiring that the surface tractions, which are implicit in $\Omega(\zeta, t)$, be correct then leads to a complicated nonlinear integrodifferential equation relating $\Omega(\sigma, t)$ to $\psi(\sigma, t)$. Rather than solving this directly, the time-dependence of $\Omega(\zeta, t)$ is adjusted to eliminate all singularities of $\psi(\zeta, t)$ on $|\zeta| \leqslant 1$.

In dimensionless form, the Kolosoff–Muskhelishvili traction equation is

$$\phi(z) + z\phi'(z)^* + \psi(z)^* = i\int [T_x(s) + T_y(s)]\, ds + \text{const} \quad (z \text{ on } \Gamma), \tag{6}$$

where s is the distance along the boundary, and $T_x(s)$ and $T_y(s)$ are the components of the tractions (the dimensionless form of the force per unit length of our cylinder, per unit length of the boundary) applied to Γ in the x and y directions. For Poisson's ratio $= \frac{1}{2}$ (incompressibility), the displacement equation is

$$\phi(z) - z\phi'(z)^* - \psi(z)^* = 2[u_x(z) + i u_y(z)] \quad (z \text{ in } (\Gamma + D)), \tag{7}$$

where $u_x(z)$ and $u_y(z)$ are the displacements in the x and y directions. In the present context these are the components of velocity. Given $\boldsymbol{T}(z)$ or $\boldsymbol{u}(z)$ on Γ, and the requirement that $\phi(z)$ and $\psi(z)$ be analytic, then the functions $\phi(z)$ and $\psi(z)$ are determined throughout D by (6) or (7): either equation constitutes a mathematically well-posed problem. Conversely, the functions $\phi(z)$ and $\psi(z)$ provide a complete description of the elastic state of the body.† Because of their physical significance, the functions on Γ are interpreted to mean – and therefore defined to be – their

† The left-hand side of (6) has a related significance within D (Muskhelishvili 1953*a*, §33.) When $\phi(z)$ and $\psi(z)$ are known in D, (7) gives the displacements throughout D, and the stresses in D are given by

$$\tau_{xx} + \tau_{yy} = 4\,\mathrm{Re}\,\phi'(z),$$

$$\tau_{yy} - \tau_{xx} + i2\tau_{xy} = 2[z^*\phi''(z) + \psi'(z)],$$

$\phi(z)$ and $\psi(z)$ must be single-valued in $(D + \Gamma)$. If either is not, then physically unacceptable discontinuities appear. In all the mapped forms of the Kolosoff–Muskhelishvili equations, it is postulated that the mapping is one-to-one (Muskhilishvili 1953*a*, §47); this is assured by the present hypotheses on Γ (Titchmarsh 1939, §6.45).

limiting values as z approaches the boundary Γ from within D. Actually, there is an arbitrariness in the functions $\phi(z)$ and $\psi(z)$, and in the constant of integration in (6), having to do with rigid-body motions (cf. Muskhelishvili 1953*a* or Sokolnikov 1956). Being concerned only with shape changes, these are of no concern. It is convenient and acceptable, however, to set $\phi(0)$ and the constant in (6) to zero.

Surface tension gives rise to a surface traction normal to the boundary and of magnitude κ. Let α be the angle between the outward normal of the boundary and the direction of the x-axis, measured counterclockwise. Then using $\kappa(s) = \mathrm{d}\alpha(s)/\mathrm{d}s$, (6) becomes

$$\phi(z)+z\phi'(z)^*+\psi(z)^* = -\mathrm{e}^{\mathrm{i}\alpha(z)} \quad (z \text{ on } \Gamma). \tag{8}$$

Now let $z = \Omega(\zeta)$ conformally map $|\zeta| \leqslant 1$ onto $(D+\Gamma)$. Let $\phi_1(\zeta) = \phi[\Omega(\zeta)]$ and $\psi_1(\zeta) = \psi[\Omega(\zeta)]$. Using the expression for the angle α at the z-plane image of a boundary point σ (Muskhelishvili 1953*a*, equation (49.3)), (8) becomes

$$\phi_1(\sigma)+\frac{\Omega(\sigma)}{\Omega'(\sigma)^*}\phi_1'(\sigma)^*+\psi_1(\sigma)^* = -\frac{\sigma\Omega'(\sigma)}{|\Omega'(\sigma)|}. \tag{9}$$

Likewise, letting $u_1(\zeta) = u[\Omega(\zeta)] = u_x+\mathrm{i}u_y$, and combining (6) and (7),

$$u_1(\sigma) = \phi_1(\sigma)+\frac{\sigma\Omega'(\sigma)}{2\,|\Omega'(\sigma)|}. \tag{10}$$

Equation (10) applies only on the boundary. As all of the following analysis occurs in the ζ-plane, the subscripts '1' in the above equations will hereafter be dropped. The functions $\phi(\zeta)$ and $\psi(\zeta)$ are to be analytic on $|\zeta| \leqslant 1$. For convenience, specify $\Omega(0,t) = 0$. This can give rise to a rigid-body motion, but again, these are of no concern.

Consider now a material point on the boundary Γ that is the z-plane image of σ at time t; that is, the point $\Omega(\sigma,t) = \Omega(\mathrm{e}^{\mathrm{i}\vartheta},t)$. Mathematically, it is assumed that the shape evolves smoothly in time: specifically that $\dot{\Omega}(\sigma,t)$ is continuous. Presumably, this is implicit in the equations of motion; physically, anything else would seem preposterous. During the time increment $\mathrm{d}t$, this point moves to the point $\Omega(\sigma,t)+u(\sigma,t)\,\mathrm{d}t$. In terms of the description we are choosing to use, this new point must be given by $\Omega[\mathrm{e}^{\mathrm{i}(\vartheta+\mathrm{d}\vartheta)},t+\mathrm{d}t]$. That is, our material point must be the z-plane image, by the mapping function at time $t+\mathrm{d}t$, of some point – not necessarily the original point – on the unit circle in the ζ-plane:

$$\begin{aligned}\Omega[\mathrm{e}^{\mathrm{i}(\vartheta+\mathrm{d}\vartheta)},t+\mathrm{d}t] &= \Omega(\mathrm{e}^{\mathrm{i}\vartheta},t)+u(\mathrm{e}^{\mathrm{i}\vartheta},t)\,\mathrm{d}t\\ &= \Omega(\mathrm{e}^{\mathrm{i}\vartheta},t)+\Omega'(\mathrm{e}^{\mathrm{i}\vartheta},t)\,\mathrm{i}\mathrm{e}^{\mathrm{i}\vartheta}\,\mathrm{d}\vartheta+\dot{\Omega}(\mathrm{e}^{\mathrm{i}\vartheta},t)\,\mathrm{d}t.\end{aligned} \tag{11}$$

The change in ϑ will depend upon both ϑ and t, so write $\mathrm{d}\vartheta = \dot{\Theta}(\sigma,t)\,\mathrm{d}t$ where $\dot{\Theta}(\sigma,t)$ is real. Combining with (10),

$$\phi(\sigma,t)+\frac{\sigma\Omega'(\sigma,t)}{2\,|\Omega'(\sigma,t)|} = \Omega'(\sigma,t)\,\mathrm{i}\sigma\dot{\Theta}(\sigma,t)+\dot{\Omega}(\sigma,t). \tag{12}$$

Rearranging,
$$\frac{\phi(\sigma,t)}{\sigma\Omega'(\sigma,t)} = -\left[\frac{1}{2\,|\Omega'(\sigma,t)|}-\mathrm{i}\dot{\Theta}(\sigma,t)\right]+\frac{\dot{\Omega}(\sigma,t)}{\sigma\Omega'(\sigma,t)}. \tag{13}$$

Now $\phi(\zeta,t)$ is required to be analytic; $\dot{\Omega}(\zeta,t)$ is analytic; by the requirement that there be no critical points on $(D+\Gamma)$, $\Omega'(\zeta,t) \neq 0$ on $|\zeta| \leqslant 1$; we have chosen $\Omega(0,t) = 0$, so $\dot{\Omega}(0,t) = 0$; and we have chosen $\phi(0) = 0$. The first and last terms are therefore boundary values of functions analytic on $|\zeta| \leqslant 1$. It follows that the two terms in brackets are the real and imaginary parts of the boundary values of some function analytic and single-valued on $|\zeta| \leqslant 1$. The real part is known, so this function is

$$F(\zeta, t) = \frac{1}{2\pi\mathrm{i}} \oint \frac{1}{2\,|\Omega'(\sigma, t)|} \frac{\sigma+\zeta}{\sigma(\sigma-\zeta)} \mathrm{d}\sigma \equiv \frac{1}{2\pi\mathrm{i}} \oint \frac{1}{|\Omega'(\sigma, t)|} \left(\frac{1}{\sigma-\zeta} - \frac{1}{2\sigma}\right) \mathrm{d}\sigma, \quad (14a, b)$$

where the contour integrals are taken around the unit circle, $|\sigma| = 1$.

The Cauchy integrals† in (14) define a function that is analytic on $|\zeta| < 1$ and on $|\zeta| > 1$ but discontinuous across $|\zeta| = 1$. As noted above, we are concerned with functions defined by Cauchy integrals and the limits on the boundary of integration as ζ approaches a boundary point σ from within the contour of integration. A Plemelj formula gives this limit as

$$F(\sigma_0, t) = \frac{1}{2\,|\Omega'(\sigma_0, t)|} + \frac{1}{2\pi\mathrm{i}} \oint \frac{1}{2\,|\Omega'(\sigma, t)|} \frac{\sigma+\sigma_0}{\sigma(\sigma-\sigma_0)} \mathrm{d}\sigma. \quad (15)$$

Singular integrals such as (15) will always be taken to mean the Cauchy principal value, but no special notation will be employed. Further, the function will be regarded as defined within and on the path of integration in just this sense. $F(\zeta, t)$, for example, as given by (14) and (15) will be regarded as defining a single function on $|\zeta| \leqslant 1$. Derivatives are obtained by differentiating (14*b*) under the integral sign. Since

$$\left(\frac{\sigma+\sigma_0}{\sigma-\sigma_0}\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{i}\sigma}\right) = \left[\frac{\mathrm{i} \sin(\vartheta_0 - \vartheta)}{1-\cos(\vartheta_0-\vartheta)}\right](\mathrm{d}\vartheta), \quad (16)$$

it is obvious that $F(\sigma, t)$ indeed meets the requirement of (13).

Upon substituting into (13), and noting that each of the resulting terms are obviously boundary values of functions analytic on $|\zeta| \leqslant 1$, one obtains the explicit formula

$$\phi(\zeta, t) = -\zeta\Omega'(\zeta, t)\, F(\zeta, t) + \dot{\Omega}(\zeta, t) \quad (|\zeta| \leqslant 1). \quad (17)$$

Differentiating and substituting into (9),

$$[-\sigma\Omega'(\sigma, t)\, F(\sigma, t) + \dot{\Omega}(\sigma, t)] + \frac{\Omega(\sigma)}{\Omega'(\sigma)^*}\left\{\frac{\mathrm{d}}{\mathrm{d}\sigma}[-\sigma\Omega'(\sigma, t)\, F(\sigma, t) + \dot{\Omega}(\sigma, t)]\right\}^* + \psi(\sigma)^* = -\frac{\sigma\Omega'(\sigma)}{|\Omega'(\sigma)|}. \quad (18)$$

Rearranging,

$$\psi(\sigma, t)^* = \sigma\Omega'(\sigma, t)\left[F(\sigma, t) - \frac{1}{|\Omega'(\sigma, t)|}\right] - \dot{\Omega}(\sigma, t) + \frac{\Omega(\sigma, t)}{\Omega'(\sigma, t)^*}\left\{\frac{\mathrm{d}}{\mathrm{d}\sigma}[\sigma\Omega'(\sigma, t)\, F(\sigma, t) - \dot{\Omega}(\sigma, t)]\right\}^*. \quad (19)$$

By (15) the first bracketed group in (19) is just $[-F(\sigma, t)^*]$. Rearranging and conjugating,

$$\Omega'(\sigma, t)\,\psi(\sigma, t) = -\Omega'(\sigma, t)\, F(\sigma, t)\,[\sigma\Omega'(\sigma, t)]^* - \Omega'(\sigma, t)\,\dot{\Omega}(\sigma, t)^* + \Omega(\sigma, t)^* \frac{\mathrm{d}}{\mathrm{d}\sigma}[\sigma\Omega'(\sigma, t)\, F(\sigma, t)] - \Omega(\sigma, t)^* \frac{\mathrm{d}}{\mathrm{d}\sigma}[\dot{\Omega}(\sigma, t)]. \quad (20)$$

† The various manipulations (Plemelj formulae, limits of derivatives, etc.) of Cauchy integrals in this article are valid for the contour ($|\zeta| = 1$) and density functions under consideration (e.g. Muskhelishvili 1953*a*, 1953*b*; Carrier *et al.* 1966). Physically, the present cases require the stresses be continuous, so $\phi''(\zeta)$ is required to be well-defined on $|\zeta| \leqslant 1$. As a consequence, it is necessary that $F'(\zeta)$ be well-defined on $|\zeta| \leqslant 1$, including as $\zeta \to \sigma$. Expressed in terms of the z-space geometry of Γ, differentiability of $\kappa(s)$ suffices (Muskhelishvili 1953*a*, §69).

The continuity of $\dot{\Omega}(\zeta,t)$ together with the existence of $\Omega^{(n)}(\zeta,t)$ implies that $\partial\Omega'(\zeta,t)/\partial t = \partial\dot{\Omega}(\zeta,t)/\partial\zeta$. Noting that $(\mathrm{d}/\mathrm{d}\sigma)[\Omega(\sigma,t)^*] = -\sigma^{-2}\Omega'(\sigma,t)^*$, equation (20) then becomes

$$\Omega'(\sigma,t)\,\psi(\sigma,t) = \frac{\mathrm{d}}{\mathrm{d}\sigma}[\sigma\Omega'(\sigma,t)\,\Omega(\sigma,t)^*F(\sigma,t)] - \frac{\mathrm{d}}{\mathrm{d}t}[\Omega'(\sigma,t)\,\Omega(\sigma,t)^*]. \tag{21}$$

This is one form of the shape-evolution equation. From it one obtains equivalently

$$\psi(\sigma,t) = \Omega(\sigma,t)^*\left[\left(1+\sigma\frac{\Omega''(\sigma,t)}{\Omega'(\sigma,t)}\right)F(\sigma,t)+\sigma F'(\sigma,t)-\frac{\dot{\Omega}'(\sigma,t)}{\Omega'(\sigma,t)}\right] - [\sigma\Omega'(\sigma,t)]^*F(\sigma,t)-\dot{\Omega}(\sigma,t)^*. \tag{22}$$

Given $\Omega(\sigma,t_1)$ – i.e. at all points of the unit circle, at one instant of time – all the quantities in (21) or (22) are known except $\psi(\sigma,t_1)$ and $\dot{\Omega}(\sigma,t_1)$. The requirement is that $\dot{\Omega}(\sigma,t_1)$ be chosen such that $\psi(\sigma,t_1)$ is the boundary value of a function analytic and single valued on $|\zeta| \leqslant 1$. This is a problem in analytic continuation. It seems that, in principle, either (21) or (22) uniquely determines $\dot{\Omega}(\zeta,t)$, except for rigid-body motions: This may be seen by considering that the original problem certainly has a solution (it is mathematically well-posed), which if known would provide ϕ and ψ; and the shape evolution can certainly be described by (an essentially unique) Ω. Equation (21) would then give $\dot{\Omega}(\sigma,t_1)$ explicitly. By the uniqueness of analytic continuation, only one particular $\dot{\Omega}(\zeta,t_1)$ could result. This applies for all $t_1 \geqslant t_i$ and so determines $\Omega(\zeta,t)$. The fact that (21) determines the shape evolution does not, however, gives us that evolution. At least, this author has no idea how to go about 'solving' (21), given an initial shape $\Omega(\sigma,t_i)$.

If a parametric description has been chosen, however, the problem simplifies. It must then be shown that the chosen parametric form satisfies (21), at least in principle; that is, that the parameters may be chosen such that (21) is indeed the boundary value of a function analytic on $|\zeta| \leqslant 1$; or alternatively stated, that the parameters may be chosen such that values on $|\zeta| = 1$ given by (21) may be continued analytically onto $|\zeta| \leqslant 1$, and that this analytic continuation is single-valued there. The values of the parameters must then actually be determined. Typically, the second task automatically fulfils the first; but sometimes the former is feasible while the latter is not. If it turns out that the parameterization does not admit a solution, this does not mean that the solution of the initial-value problem for the chosen $\Omega(\sigma,t_i)$ does not exist: it certainly does. It is just that the parameterization does not give the time evolution, i.e. the guess was wrong. All these remarks apply equally to (22). To summarize:

Shape evolution

Given an initial shape bounded by the (simple, smooth, closed) curve $z = \Omega(\sigma,t_i)$, the requirement that $\psi(\zeta,t)$ be a function analytic and single-valued on $|\zeta| \leqslant 1$ for $t \geqslant t_i$, and having boundary values $\psi(\sigma,t)$ given by (21) or (22), determines uniquely $\dot{\Omega}(\sigma,t)$ and therefore $\Omega(\zeta,t)$.

3. General observations

Equations (21) or (22), giving the shape evolution, are fairly general and apply to any $\Omega(\zeta,t)$ describing a region bounded by a simple smooth closed curve. The dynamics of the flow is unaffected by rigid-body motions, since inertia is neglected and there are no external fields. Thus the location and orientation of the shape is arbitrary. It is often helpful to use this arbitrariness to cast $\Omega(\zeta, t)$ in a convenient form. In addition, it is also often convenient to use a scaling factor as

$$\Omega(\zeta, t) = B(t)\,\omega(\zeta, t), \tag{23}$$

where $B(t)$ is real and positive. Convenient properties may then be imposed on $\omega(\zeta, t)$. Having normalized the spatial scale of the z-space region (i.e. the distance R_0) and chosen $\omega(\zeta, t)$, the time-dependence of $B(t)$ is determined from the time-dependence of $\omega(\zeta, t)$ through the requirement of area conservation. The area enclosed by $z = \Omega(\sigma, \lambda)$ is given by any of the standard formulae

$$\text{area} = \int_0^1 \mathrm{d}\rho\,\rho \int_{-\pi}^{\pi} \mathrm{d}\vartheta\, |\Omega'(\rho\, \mathrm{e}^{\mathrm{i}\vartheta}, t)|^2 \tag{24a}$$

$$= \frac{1}{2\mathrm{i}} \oint \Omega(\sigma, t)^* \, \Omega'(\sigma, t)\, \mathrm{d}\sigma \tag{24b}$$

$$= \pi \sum_{n=1}^{\infty} \frac{|\Omega^{(n)}(0, t)|^2}{n!(n-1)!}. \tag{24c}$$

If we write the Taylor series of $\Omega(\zeta, t)$ as

$$\Omega(\zeta, t) = \sum_{n=0}^{\infty} a_n(t)\, \zeta^n \quad (|\zeta| \leqslant 1), \tag{25}$$

then (24*c*) becomes

$$\text{area} = \pi \sum_{n=1}^{\infty} n\,|a_n(t)|^2. \tag{24d}$$

Incidentally, the requirement of area conservation is contained in (21): Integrating around $|\sigma| = 1$, the left-hand side vanishes by the Cauchy–Goursat theorem; the first term on the right-hand side vanishes directly; and so the integral of the last term, which by (24*b*) is proportional to the time derivative of the area, equals zero.

Let us recall a few elementary properties of conformal maps. First, if $\Omega'(\zeta_0) = 0$ or ∞, then the z-plane image of a smooth ζ-plane curve through the point ζ_0 will have a discontinuous change of direction. This applies in particular to the curve $|\zeta| = 1$. Suppose $\Omega'(\zeta) \sim (\zeta - \zeta_0)^\alpha$ near ζ_0. If $|\zeta_0| = 1$, then the angle of the change depends on α as follows: ($\alpha = 1$) gives an inward-pointing cusp; ($0 < \alpha < 1$) gives an inward pointing corner of finite angle; ($-1 < \alpha < 0$) gives an outward-pointing corner; and ($\alpha = -1$) gives an outward-pointing cusp. If the point ζ_0 lies close to but outside $|\zeta| = 1$, then the z-plane image of $|\zeta| = 1$ will exhibit a somewhat rounded-off version of the singularity. The inward-pointing cusps, and their rounded-off versions, are obviously a ubiquitous feature of polynomial maps. It may also be noted that two zeros of $\Omega'(\zeta)$ create a loop in $\Omega(\zeta)$ if they are too close to each other and to $|\zeta| = 1$. Another common singularity is a simple pole. Suppose $\Omega(\zeta) \sim (\zeta - \zeta_0)^{-1}$ near ζ_0. This locally approximates a bilinear transformation and therefore locally maps a circular arc into another circular arc; so if ζ_0 lies near $|\zeta| = 1$, the z-image of this curve near ζ_0 is a large approximately circular arc.

Once $\Omega(\zeta, t)$ has been found, by whatever means, (17) gives $\phi(\zeta, t)$; and using Cauchy's integral formula, (21) or (22) gives $\psi(\zeta, t)$. Then the internal velocities are given by (7) and the internal pressure and stresses, by the formulae in the footnote on p. 00. If only the surface velocities are required, (10) provides them without the need to evaluate $\psi(\zeta, t)$. Though straightforward, these computations are typically complicated and tiresome (Hopper 1990*a*), and yield complicated formulae.

Three conjectures dealing with important forms of mapping functions will now be given. They seem likely to be always or almost always true, but proving so depends on the existence of a solution to a system of ordinary differential equations, which the author has been unable to demonstrate. The conjectures may be regarded as guides for guessing suitably parameterized mapping functions, their correctness to be verified subsequently.

Polynomial Maps (conjecture)

A shape given initially by an Nth-order polynomial map will be described by an Nth-order polynomial map at all subsequent times; that is, if $\Omega(\zeta, t_1) = P_N(\zeta, t_1)$, then $\Omega(\zeta, t) = P_N(\zeta, t)$ for $t > t_1$, where of course $P_N(\zeta, t)$ is an Nth-order polynomial in ζ having t-dependent coefficients. The argument is simple: let

$$P_N(\zeta, t) = \sum_{n=1}^{N} a_n(t)\, \zeta^n. \tag{26}$$

There are $2N$ time-dependent parameters, $\operatorname{Re} a_n(t)$ and $\operatorname{Im} a_n(t)$. We have

$$\Omega_N(\sigma, t)^* = \sum_{n=1}^{N} a_n(t)^* \sigma^{-n}. \tag{27}$$

Then both terms on the right-hand side of (21) include terms in $\sigma^{-n}, n \leqslant N$. The analytic continuation of (21) onto $|\zeta| < 1$ is obtained by simply replacing σ by ζ. Since $\Omega'(0, t) \neq 0$, it is necessary that the coefficients of each of the σ^{-n} terms on the right-hand side of (21) vanish if $\psi(\zeta, t)$ is not to have a pole at $\zeta = 0$. The real and imaginary parts of these coefficients lead to $2N$ algebraic equations containing $2N$ unknowns, namely $\operatorname{Re} \dot{a}_n(t)$ and $\operatorname{Im} \dot{a}_n(t)$. In principle, solving these equations provides $2N$ coupled first-order ordinary differential equations of such forms as $\operatorname{Re} \dot{a}_n(t) = g_n[\operatorname{Re} a_1(t), \operatorname{Re} a_2(t), \ldots, \operatorname{Re} a_N(t), \operatorname{Im} a_1(t), \operatorname{Im} a_2(t), \ldots, \operatorname{Im} a_N(t)]$, and the initial values are known. The conclusion would then follow from the existence of a solution to this system of equations, but this is not easily proved. The difficulty in doing so arises because the functions g_n are determined by the solution of a system of algebraic equations, and the properties of the g_n are not *a priori* obvious.† The 'fact' (which was argued but not proved at the end of §2) that (21) is sufficient to determine uniquely the function $\Omega(\zeta, t)$, given $\Omega(\zeta, t_1)$, implies nothing about a hypothesized parameterization of $\Omega(\zeta, t)$; if it happened that in some case there were no solution to the system of equations for the $a_n(t)$, this would imply only that the

† Standard theorems state conditions on the functions g_n (briefly, that they be continuous, single-valued, and satisfy a Lipshitz condition) sufficient to ensure the existence of a (unique) solution (e.g. Ince 1926). Requiring that the coefficients of each of the σ^{-n} terms on the right-hand side of (21) vanish leads to equations of the form

$$\frac{d}{dt}[b_n(a_1, a_2, \ldots, a_N] = h_n(a_1, a_2, \ldots, a_N),$$

where
$$b_n = \sum_{m=1}^{N+1-n} m a_m a^*_{m+n-1},$$

and where the functions h_n are given by a triple sum whose terms involve the coefficients a_m both directly and in factors $F^{(k)}(0)$. Before the standard theorems can be used, either the coefficients a_n must be expressed in terms of the b_n and substituted into the right-hand side of the first of these; or the $\dot{a}_n(t)$ must be expressed in terms of the $\dot{b}_n(t)$ and the system replaced with a system of the form $\dot{a}_n(t) = g_n(a_1, a_2, \ldots, a_N)$; or at least certain properties of the resulting equations must be deduced. An unsolved example is given in §4.3.

parameterization is unacceptable. On the other hand, the argument at the close of §2 does imply that if a solution in that parameterization exists, then it is unique.

Obtaining analytical solutions for the $a_n(t)$ is feasible only in the simplest cases. Numerical solution of the system does not appear to present any special difficulties. That $a_n(0) = 0$ implies nothing in particular about $a_n(t)\,(1 < n < N)$. In particular cases, however, symmetry may imply $a_n(t) = 0$ for certain n.

Partial-Fraction Maps (conjecture)

A shape given initially by a mapping in the form of a partial-fraction expansion, i.e.

$$\Omega(\zeta, t) = \zeta \sum_{n=1}^{N} \frac{A_n(t)}{1-\alpha_n(t)\,\zeta}, \tag{28}$$

where $|\alpha_n(t)| < 1$ (and N is finite), will be described by this same form at all subsequent times. The demonstration is elementary but somewhat involved: The analytic continuation of (22) onto $|\zeta| < 1$ is obtained by simply replacing σ by ζ. Noting that

$$\Omega(\sigma)^* = \sum_n \frac{A_n^*}{\sigma-\alpha_n^*},$$

etc., it is seen that factors not involving conjugates are analytic throughout this domain, while the factors that do involve conjugation introduce, potentially, first- and second-order poles at the points $\zeta = \alpha_n(t)^*$. The adjustable parameters of the mapping must be chosen to cancel these singularities.

Substituting (28) into (22) leads to

$$\psi(\zeta) = \left(\sum_{n=1}^{N} \frac{A_n^*}{\zeta-\alpha_n^*}\right)\left\{\left[1+\frac{\zeta\Omega''(\zeta)}{\Omega'(\zeta)}\right]F(\zeta)+\zeta F'(\zeta)-\frac{\dot{\Omega}(\zeta)}{\Omega'(\zeta)}\right\}$$
$$-\left[\sum_{n=1}^{N} \frac{A_n^*}{(\zeta-\alpha_n^*)^2}\right]\zeta F(\zeta) - \sum_{n=1}^{N} \frac{\dot{A}_n^*}{\zeta-\alpha_n^*} - \sum_{n=1}^{N} \frac{A_n^*\,\dot{\alpha}_n^*}{(\zeta-\alpha_n^*)^2}. \tag{29}$$

Expanding the coefficients of $(\zeta-\alpha_n^*)^{-1}$ and $(\zeta-\alpha_n^*)^{-2}$ in a Taylor series about $\zeta = \alpha_n^*$, collecting the terms in $(\zeta-\alpha_n^*)^{-2}$, and requiring them to vanish,

$$\frac{\dot{\alpha}_n}{\alpha_n} = -F(\alpha_n^*)^*. \tag{30}$$

Similarly collecting terms in $(\zeta-\alpha_n^*)^{-1}$, one obtains

$$\frac{\dot{A}_n}{A_n} = -\left[\frac{\alpha_n^*\,\Omega''(\alpha_n^*)}{\Omega'(\alpha_n^*)}\left(\frac{\dot{\alpha}_n}{\alpha_n}\right)^* + \frac{\dot{\Omega}'(\alpha_n^*)}{\Omega'(\alpha_n^*)}\right]^*. \tag{31}$$

$F(\zeta) = F(\zeta; \alpha_1, \alpha_2, \ldots, \alpha_N, A_1, A_2, \ldots, A_N,)$ is of course well behaved at $\zeta = \alpha_n^*$ because $F(\zeta)$ is analytic on $|\zeta| \leqslant 1$. Likewise for Ω and its derivatives. The real and imaginary parts of (30) and (31) provide $4N$ coupled ordinary differential equations for the $4N$ unknowns. As before, our assertion would follow from the existence of a solution to this system of equations, but this existence has not been demonstrated. Comments similar to those regarding the polynomials apply.

Rational Maps (conjecture)

A shape given initially by a mapping that is the ratio of an Mth- to an Nth-order polynomial, with $M \leqslant N$ and having no common or repeated factors, will be described by this same for at all subsequent times; that is, if $\Omega(\zeta, t_i) = \zeta P_{M-1}(\zeta, t_i)/Q_N(\zeta, t_i)$, then $\Omega(\zeta,t) = \zeta P_{M-1}(\zeta,t)/Q_N(\zeta,t)$ for $t > t_i$, where the notation is obvious. This is almost, but not quite, a corollary of the conjecture on partial-fraction expansions: Because $P_{M-1}(\zeta,t)/Q_N(\zeta,t)$ is a proper rational function, it may be expressed in the form of (28). It follows from the conjecture on partial-fraction expansions that $\Omega(\zeta, t)$ will evolve as a rational function whose denominator is an Nth-order polynomial. The numerator will remain of order M except possibly instantaneously. The instantaneous changes of the order would occur only by a coincidental vanishing of a sum whose form depends on M and N. For example, if $M = N-1$, then the numerator would change order only if $\sum_n A_n/\alpha_n$ vanished.

What happened in these cases it that critical points of the mapping function $\Omega(\zeta, t)$, which lie on $|\zeta| > 1$, are moved onto $|\zeta| < 1$ by the conjugation. This is quite general: a given $\Omega(\zeta,t)$ having the properties we have required will represent a mapping with critical points on $|\zeta| > 1$. These might be zeros of $\Omega'(\zeta,t)$, poles, or branch points. If $\Omega(\sigma,t)$ were known, its analytic continuation would of course provide $\Omega(\zeta,t)$. The function $\Omega(\sigma,t)^*$ is well-defined and single-valued on $|\sigma| = 1$ and may be continued analytically onto some extensive region of ζ-space. It will be found that this analytic continuation† is analytic and single-valued on $|\zeta| > 1$, but that it will have singularities, or zeros of its ζ-derivative, on $|\zeta| < 1$ corresponding to the critical points of $\Omega(\zeta,t)$ located on $|\zeta| > 1$.

The fact that polynomials and this class of rational mappings can in principle be solved, is of theoretical interest in that any region of the type under consideration can be approximated to arbitrary accuracy by them (Rudin 1966). This is not particularly helpful in practice. In particular, equations describing the dynamics of the critical points are inconvenient.

4. Examples

With one exception, the following examples have no particular practical application but serve to display the simple techniques used to solve the easy cases and the origins of the difficulties encountered in the hard ones.

4.1. *Epitrochoids – binomials*

The simplest non-trivial polynomial map is specified by $\omega_1'(\zeta) = 1-\zeta$. Then $z = \omega_1(\zeta) = \zeta - \frac{1}{2}\zeta^2, |\zeta| = 1$, generates a cardioid with its cusp on the Re z-axis. Noting that $\omega_N'(\zeta) = 1-\zeta^N$ has its zeros spaced equally around the unit circle, the map

$$z = \omega_N(\zeta) = \zeta - \zeta^{N+1}/(N+1) \tag{32}$$

($|\zeta| = 1$) will have N inward-pointing cusps. It is easily shown to be an N-cusped, N-lobed regular epicycloid. (The nephroid of §1 is the $N = 2$ case.) The map for $|\zeta| =$ const > 1 has loops, while that for $|\zeta| =$ const < 1 has rounded cusps. These are all

† The analytic continuation of $\Omega(\sigma,t)^*$ is often denoted $\bar{\Omega}(\zeta,t)$. Thus, $\bar{\Omega}(\sigma,t) = \Omega(\sigma,t)^*$. Using the notation of (25) for the Taylor series of $\Omega(\zeta,t)$,

$$\bar{\Omega}(\zeta,t) = \sum_{n=0}^{\infty} a_n(t)^* \zeta^{-n}.$$

At first glance, this form suggests that any conjectured parameterization might be tested with a Laurent series about $\zeta = 0$, and that the shape-evolution equations could usefully be expressed in such terms. The equation does not, however, identify all of the singularities on $|\zeta| \leqslant 1$ in any way that would be obvious, and integrating around $|\zeta| = 1$ does not necessarily give the Laurent coefficients.

epitrochoids, but the theory applies only to the latter case. From §3, the mapping function of an N-symmetric epitrochoid is expected to evolve as a polynomial of order $N+1$; and so its derivative will have N zeros, which are critical points of the map. Physically, the latent cusps must get smoother with time, and the symmetry will not change; so the only reasonable expectation is that the zeros of $\omega'_N(\zeta)$ will move outward radially. Hence, the obvious parameterization is given by (2). (The appearance of terms in $\zeta^n, 1 < n \leqslant N$, would destroy the symmetry.) The zeros of $\Omega'_N(\zeta, t)$ are located at $\lambda^{-1}\,\mathrm{e}^{\mathrm{i}(n-1)2\pi/N}$, $n = 1, 2, \ldots, N$. Clearly, $\lambda(0) = 1$, $\lambda(\infty) = 0$, $\dot{\lambda}(t) < 0$. Fixing the area at π, and using (24d),

$$B_N(\lambda) = \lambda^{-1}[1+\lambda^{2N}/(N+1)]^{-\frac{1}{2}}. \tag{33}$$

Condensing the notation,

$$\Omega(\zeta) = \lambda B\left(\zeta - \frac{\lambda^N}{N+1}\zeta^{N+1}\right) \tag{34a}$$

$$= \left(1+\frac{\lambda^{2N}}{N+1}\right)^{-\frac{1}{2}}\left(\zeta - \frac{\lambda^N}{N+1}\zeta^{N+1}\right). \tag{34b}$$

Note that if the quantitative time dependence is not required, but only the sequence of shapes, the problem is solved.

For the time evolution, we substitute into (21). As we knew would be the case, the poles are avoidable. Equation (21) becomes

$$\Omega'(\sigma)\,\psi(\sigma) = \lambda^2B^2\frac{\mathrm{d}}{\mathrm{d}\sigma}\left[\left(1+\frac{\lambda^{2N}}{N+1} - \lambda^N\sigma^N - \frac{\lambda^N}{N+1}\sigma^{-N}\right)F(\sigma)\right] - \frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma^{-1} - \lambda^{N+2}B^2\sigma^{N-1} - \frac{\lambda^{N+2}B^2}{N+1}\sigma^{-N-1}\right). \tag{35}$$

Carrying out the differentiations, obtaining the analytic continuation by replacing σ with ζ, and ignoring those terms that are obviously analytic throughout $|\zeta| \leqslant 1$,

$$\Omega'(\zeta)\,\psi(\zeta) = \lambda^2B^2\left[\frac{N\lambda^N}{N+1}\zeta^{-N-1}F(\zeta) - \frac{\lambda^N}{N+1}\zeta^{-N}F'(\zeta)\right] + \frac{1}{N+1}\left[(N+2)\,\lambda^{N+1}B^2 + 2\lambda^{N+2}BB'\right]\dot{\lambda}\zeta^{-N-1} + (\text{analytic terms}). \tag{36}$$

Expanding $F(\zeta)$ and $F'(\zeta)$ in their Taylor series about $\zeta = 0$,

$$\begin{aligned}(N+1)\,(\lambda^{N+2}B^2)^{-1}\,\Omega'(\zeta)\,\psi(\zeta) &= \zeta^{-N-1}[NF(0) + (N+2+2\lambda B'/B)\,(\dot{\lambda}/\lambda)] \\ &+ \zeta^{-N}F'(0)\,(N-1) + \zeta^{-N+1}F''(0)\left(\frac{N}{2!}-1\right) + \zeta^{-N+2}F'''(0)\left(\frac{N}{3!}-\frac{1}{2!}\right) + \ldots \\ &+ \zeta^{-2}F^{(N-1)}(0)\left[\frac{N}{(N-1)!} - \frac{1}{(N-2)!}\right] + (\text{analytic terms}).\end{aligned} \tag{37}$$

Differentiating (14b)

$$F^{(n)}(0) = \frac{n!}{2\pi\mathrm{i}}\oint[\lambda^2B^2(1-\lambda^N\sigma^N)\,(1-\lambda^N\sigma^{-N})]^{-\frac{1}{2}}\sigma^{-(n+1)}\,\mathrm{d}\sigma \tag{38a}$$

$$= \frac{n!}{2\pi\lambda B}\int_{-\pi}^{\pi}\frac{\cos n\vartheta - \mathrm{i}\sin n\vartheta}{(1-2\lambda^N\cos N\vartheta + \lambda^{2N})^{\frac{1}{2}}}\mathrm{d}\vartheta \quad (n \geqslant 1). \tag{38b}$$

The $\sin n\vartheta$ integral vanishes by its oddness. Expanding $(1-(2\lambda^N/1+\lambda^{2N})\cos N\vartheta)^{-\frac{1}{2}}$ in a (uniformly convergent) power series, $F^{(n)}(0)$ may be expressed as an infinite series, each term of which contains an integral of the form

$$\int_0^\pi \cos n\vartheta \cos^m N\vartheta \,\mathrm{d}\vartheta \quad (m > 0). \tag{39}$$

Every such term can be reduced to integration of a single cosine factor by

$$\begin{aligned}\cos n\vartheta \cos^m N\vartheta &= (\cos N\vartheta \cos n\vartheta)\cos^{m-1} N\vartheta \\ &= \tfrac{1}{2}\{[\cos(N+n)\vartheta + \cos(N-n)\vartheta]\cos N\vartheta\}\cos^{m-2} N\vartheta \\ &= \dots\end{aligned} \tag{40}$$

For $n < N$, each integral vanishes. Hence $F^{(n)}(0) = 0$ for $1 \leqslant n < N$.

Returning to (37), and recalling that $\Omega'(0) \neq 0, \psi(\zeta)$ will be non-singular at the origin if and only if

$$(N+2+2\lambda B'/B)\,\dot{\lambda} = -N\lambda F(0). \tag{41}$$

Analogous to the preceding, (14) leads to

$$F(0) = \frac{1}{2\pi\lambda B}\int_0^{\pi/2} (1-2\lambda^N \cos N\vartheta + \lambda^{2N})^{-\frac{1}{2}}\,\mathrm{d}\vartheta. \tag{42}$$

Changing variables, using periodicity, and referring to Gradshteyn & Rhyzhik (1980, equation (3.674-1)),

$$F(0) = \frac{1}{\pi\lambda B}K(\lambda^N), \tag{43}$$

where $K(k)$ is the complete elliptic integral of the first kind defined by (conventions vary)

$$K(k) = \int_0^\pi (1-k^2\sin^2\phi)^{-\frac{1}{2}}\,\mathrm{d}\phi. \tag{44}$$

Using (33) and (41), (43) becomes

$$\left(\frac{1-\dfrac{\lambda^{2N}}{N+1}}{1+\dfrac{\lambda^{2N}}{N+1}}\right)\dot{\lambda} = -\frac{\lambda}{\pi}\left(1+\frac{\lambda^{2N}}{N+1}\right)^{\frac{1}{2}} K(\lambda^N). \tag{45}$$

Rearranging, integrating, changing the dummy variable of the integral to $k = \lambda^N$, and replacing the parameter λ with $\nu = \lambda^N$, we have finally

$$t = \frac{\pi}{N}\int_\nu^1 \left(1-\frac{k^2}{N+1}\right)\left(1+\frac{k^2}{N+1}\right)^{-\frac{3}{2}} [kK(k)]^{-1}\,\mathrm{d}k, \tag{46}$$

$$\Omega(\zeta) = \left(1+\frac{\nu^2}{N+1}\right)^{-\frac{1}{2}}\left(\zeta - \frac{\nu}{N+1}\zeta^{N+1}\right). \tag{47}$$

Equations (46) and (47) provide a complete description of the evolution of the epitrochoids. As forecast, an epitrochoid evolves through a continuous sequence of other epitrochoids of the same symmetry and area. The rolling circles generating the sequence change diameters with ν, though their ratios remain fixed. The limiting behaviour as $t \to \infty$ gives the decay constant for the Nth-order, small-amplitude harmonic distortion of a cylinder.† Numerical quadrature of (46), and obtaining approximations valid for very short and for very long times, is straightforward. The results for $N=2$ were included in figure 1. Some quantitative details of cusp rounding are given in Hopper (1990*b*). In addition to the epitrochoids, several other elementary maps lead to time integrals involving a factor $K(k)^{-1}$. This arises merely from $|\Omega'(\sigma,t)|$ having a simple form leading naturally to an elliptic integral and has no particular significance.

4.2. *Trochoids – a limiting case*

As N becomes large, the epitrochoids become a circle with tiny surface ripples locally approximating a trochoid. Renormalizing the spatial scale and taking the limit as $N\to\infty$ gives the flattening of viscous half-space bounded by a trochoid. In dimensional form, the epitrochoid Γ_0 bounded an area of πR_0^2. The approximate distance between radial minima – the latent cusps – was $2\pi R_0/N$. It is convenient to renormalize using the distance scale $R_\infty = R_0/N$. (That is, in all the dimensionless forms at the start of §2, replace R_0 by R_∞.) Then the dimensional parametric forms of (46) and (47) are

$$X_0(\vartheta,\nu) = NR_\infty\left(1+\frac{\nu^2}{N+1}\right)^{\frac{1}{2}}\left[\cos\vartheta-\frac{\nu}{N+1}\cos(N+1)\vartheta\right], \tag{48a}$$

$$Y_0(\vartheta,\nu) = NR_\infty\left(1+\frac{\nu^2}{N+1}\right)^{\frac{1}{2}}\left[\sin\vartheta-\frac{\nu}{N+1}\sin(N+1)\vartheta\right], \tag{48b}$$

$$t_0 = \frac{\eta R_\infty}{\gamma}\pi\int_\nu^1\left(1-\frac{k^2}{N+1}\right)\left(1+\frac{k^2}{N+1}\right)^{-\frac{3}{2}}[kK(k)]^{-1}\,\mathrm{d}k. \tag{49}$$

Let $\phi = N\vartheta$, rotate the figure by interchanging X_0 and Y_0 so that the fluid occupies the lower half of the plane, take the limit at $N\to\infty$, and translate so that the net area of fluid above $Y_0=0$ is zero. One obtains

$$X(\phi,\nu) = X_0(\phi,\nu)/R_\infty = \phi-\nu\sin\phi, \tag{50a}$$

$$Y(\phi,\nu) = Y_0(\phi,\nu)/R_\infty = -(\tfrac{1}{2}\nu^2+\nu\cos\phi), \tag{50b}$$

$$t(\nu) = \frac{\gamma t_0}{hR_\infty} = \pi\int_\nu^1[kK(k)]^{-1}\,\mathrm{d}k \quad (0<\nu<1). \tag{51}$$

The interval $0\leqslant\phi\leqslant\pi$ describes a half-period of the curve. Equations (50) describe a trochoid that is the locus of a point on a disk of radius R_∞, located a distance νR_∞ from its centre, when the disk rolls on $Y_0 = -R_\infty(1+\frac{1}{2}\nu^2)$. The depth is $Y(\pi,\nu)-Y(0,\nu)=2\nu$. The singular case ($\nu=1$) is a cycloid. As $\nu\to 0$, the trochoid approaches the sinusoidal form $Y=\nu\cos X$. Typical cases are shown in figure 2.

The computation of $t(\nu)$ for the trochoids is typical. It is generally easiest simply to integrate (51) numerically using some standard method. The author used a polynomial approximation for $K(k)$ (Abramowitz & Stegun 1965 §17.3) and the quadrature feature built into an ordinary programmable pocket calculator (Hewlett–Packard HP-15C) to obtain values of $t(\nu)$ accurate to four significant

† If we consider a single mode of order N, with an area of πR_0^2, the distorted cylinder may be described in z-plane polar coordinates by $R(\theta,t) = [R_0^2-\frac{1}{2}A_N(t)^2]^{\frac{1}{2}}+A_N(t)\cos N\theta$. Being careful not to confuse θ in the z-plane with ϑ in the ζ-plane, a comparison with the N-lobed epitrochoid as $\nu\to 0$ leads to the correspondence, to first order in ν and in A_N, of $A_N = 2R_0\nu/(N+1)$. Then using (46), we find the small-ν limiting behaviour as $\dot{A}_N/A_N = \dot{\nu}/\nu = \frac{1}{2}N$. Thus the dimensional time constant for the exponential decay is $2\eta R_0/\gamma N$.

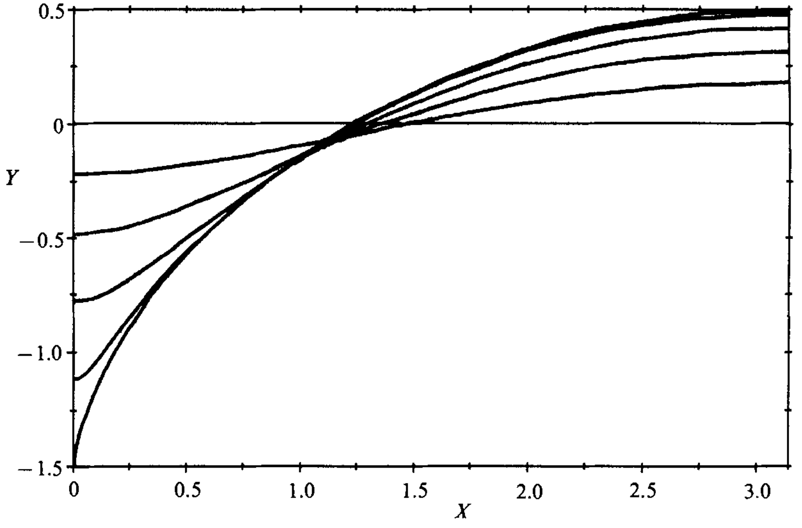

FIGURE 2. Trochoids (§4.2): $\nu = 1$ ($t = 0$), $\nu = 0.8$ ($t \approx 0.124$), $\nu = 0.6$ ($t \approx 0.789$), $\nu = 0.4$ ($t \approx 1.545$), $\nu = 0.2$ ($t \approx 2.460$), $\nu = 0$ ($t = \infty$).

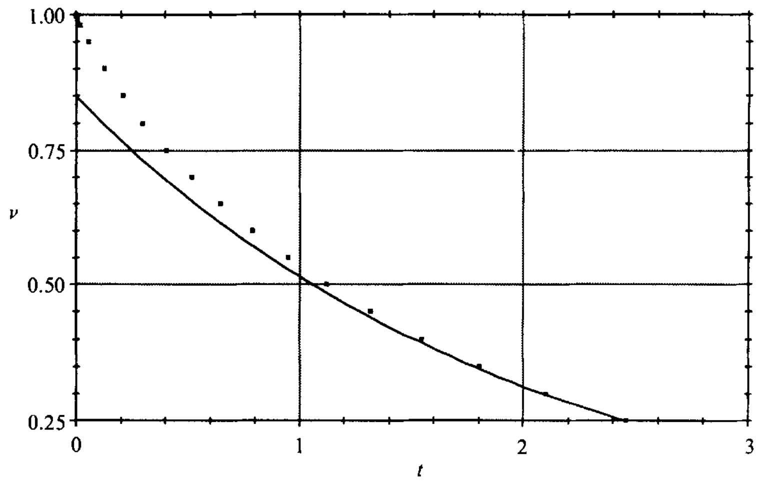

FIGURE 3. Time-dependence of the trochoids: points are $\nu(t)$ determined from (51) numerically; solid curve is the long-time approximation given by (52).

figures in less than a minute each. The author used an analytic approximation to deal with the small-t singularity, obtaining $t(\nu = 0.9997) = 1.696\,20 \times 10^{-4}$. Given the accuracy of direct quadrature, which gave $1.696\,23 \times 10^{-4}$, such procedures are not worth the effort. For long times, the $k \to 0$ expansion can be used to obtain

$$\nu \approx 0.8483\,\mathrm{e}^{-\frac{1}{2}t} \quad (t \to \infty), \tag{52}$$

where the prefactor was determined numerically. $t(\nu)$ is shown in figure 3. As $\nu \to 0$, we have a sinusoidal surface of wavelength $2\pi R_\infty$ and amplitude νR_∞ decaying as $\mathrm{e}^{-\frac{1}{2}t} = \mathrm{e}^{-\gamma t_0/2\eta R_\infty}$.

By comparison, a similar wave under the influence of gravity decays exponentially with a time constant $2\eta/\rho g R_\infty$ (e.g. Lamb 1932 §349). The ratio of the characteristic decay time for capillarity to that for gravity is thus $\rho g R_\infty^2/\gamma$. This measure of the importance of gravity relative to that of capillarity is just the Bond number defined formally in (4). In this case, and typically, it is at long times that gravity has its greatest influence. For $\gamma \approx 100$ dyn dm^{-1}, $\rho \approx 1$ g cm^{-3} and $R_\infty \approx 1$ mm (wavelength ≈ 3 mm), we have $\rho g R_\infty^2/\gamma \approx 0.1$. Note that this difference of a factor of ten occurs exponentially.

4.3. *Symmetrical cubic maps*

Though obtaining the time dependence involved a bit of effort, the shapes encountered in the evolution of an epitrochoid – given by the radial motion of the critical points – were deduced virtually without calculation. Ordinarily this is not easily done, as is demonstrated by the apparently simple case of a mapping that is symmetric across a line and is described by a cubic. Here there are only two critical points, and these must move about in the complex ζ-plane. But even the qualitative features of that motion are hard to infer.

Describe the map by the cubic

$$\Omega(\zeta, t) = a_1(t)\,\zeta + a_2(t)\,\zeta^2 + a_3(t)\,\zeta^3. \tag{53}$$

Because of the symmetry, the $a_n(t)$ can be chosen real. Substituting $\Omega(\sigma, t)$ from (53) into (21), setting the factors of σ^{-1}, σ^{-2} and σ^{-3} to zero, and requiring the area to be π, one obtains

$$a_1^2 + 2a_2^2 + 3a_3^2 = 1, \tag{54a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[a_2(a_1+2a_3)] = -[a_2(a_1+2a_3)]\,F(0; a_1, a_2, a_3) - a_1 a_3 F'(0; a_1, a_2, a_3), \tag{54b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_1 a_3) = -2a_1 a_3 F(0; a_1, a_2, a_3), \tag{54c}$$

respectively. One of the $a_n(t)$ can be eliminated with (54a) (which is just the area conservation), leaving two coupled nonlinear ordinary differential equations determining the other two.

It is intuitively appealing to express the mapping in terms of its critical points. Doing so gives mathematically natural forms for $F(0)$ and $F'(0)$ but proves otherwise inconvenient. Mathematically, the most natural formulation is to use the groupings of (54) [i.e. $b_1 = a_1 a_3$, $b_2 = a_2(a_1+2a_3)$]. Numerically, there is then no particular problem. These equations illustrate the complexity of proving the conjectures of §3 (cf. footnote on p. 00).

4.4. *Rosettes and equal circles – rational functions*

Consider the mapping

$$\omega_N(\zeta) = \zeta[1+\zeta^N/(N-1)]^{-1} \quad (|\zeta| = 1, \quad N \geqslant 2). \tag{55}$$

For $N \geqslant 3$, this describes a sort of rosette shape reminiscent of an N-cusped epicycloid, except that the lobes are more circular, especially for smaller N. As $N \to \infty$, the rosettes approach the epicycloids. For $N = 2$, the lobes of the z-image of $|\zeta| = 1$ are of infinite radius; if the figure is scaled to a finite area, as we shall do, they are found to be circular. The zeros of $\omega_N'(\zeta)$ lie on $|\zeta| = 1$, while the poles of $\omega_N(\zeta)$ lie on a circle $|\zeta| = (N-1)^{1/N} \geqslant 1$, the equality holding only for $N = 2$.

As with the epicycloids, the results of §3 combined with the symmetry suggest the parameterization of (2), resulting in

$$\Omega_N(\zeta, t) = B\lambda\zeta[1+\lambda^N\zeta^N/(N-1)]^{-1}, \tag{56}$$

$$\Omega_N'(\zeta, t) = B\lambda[1-\lambda^N\zeta^N][1+\lambda^N\zeta^N/(N-1)]^{-2}. \tag{57}$$

As (56) can obviously be expanded in partial fractions of the form of (28), and since the symmetry surely is maintained, it may tentatively be assumed that (56) is a valid parameterization of $z = \Omega_N(\sigma, t)$. This assumption is justified later. Choosing the area to be π, (24*b*) leads to

$$B_N(t) = B_N[\lambda(t)] = \frac{1-\lambda^{2N}/(N-1)^2}{\lambda(1+\lambda^{2N}/(N-1))^{\frac{1}{2}}}. \tag{58}$$

In the limit as $N \to \infty$, (56) gives again the trochoid. For $N = 2$, the non-singular cases $\lambda < 1$ are inverse ellipses of area π. As $\lambda \to 1$, the singular case of two touching circles is approached. The $N = 2$ case therefore solves the problem of the coalescence of two equal cylinders in plane flow.

Having the deduced form of $\Omega_N(\zeta, t)$, the time dependence requires only the determination of $\lambda(t)$. It will suffice to apply (30) to any one of the α_n of the partial-fraction decomposition. Noting the roots of $[1+\lambda^N\zeta^N/(N-1)]$, we have $\alpha_n = \alpha_n(\lambda) = \lambda(N-1)^{-1/N}\,\mathrm{e}^{-\mathrm{i}\pi(2n-1)/N}$ $(n = 1, 2, \ldots, N)$. Thus, in (30), $\dot{\alpha}_n/\alpha_n = \dot{\lambda}/\lambda$ for all n. From (14),

$$F_N(\alpha_N^*) = \frac{1}{2\pi\lambda B}\int_{-\pi}^{\pi} \frac{1+2\lambda(N-1)^{-1}\cos N\vartheta + [\lambda(N-1)^{-1}]^2}{(1-2\lambda^N\cos N\vartheta+\lambda^{2N})^{\frac{1}{2}}}$$

$$\times \frac{1-[\lambda(N-1)^{-1/N}]^2 - \mathrm{i}2\lambda(N-1)^{-1/N}\sin[\vartheta-(2n-1)\pi/N]}{1-[\lambda(N-1)^{-1/N}]^2-2\lambda(N-1)^{-1/N}\cos[\vartheta-(2n-1)\pi/N]}\,\mathrm{d}\vartheta. \tag{59}$$

Changing variables with $[\vartheta-(2n-1)\pi/N] \to \vartheta$,

$$F_N(\alpha_n^*, t) = \frac{1}{4\pi\lambda B}\int_{\left(-1+\frac{2n-1}{N}\right)\pi}^{\left(1+\frac{2n-1}{N}\right)\pi} \frac{1-2[\lambda(N-1)^{-1/N}]^N\cos N\vartheta+[\lambda(N-1)^{-1/N}]^{2N}}{(1+2\lambda^N\cos N\vartheta+\lambda^{2N})^{\frac{1}{2}}}$$

$$\times \frac{1-[\lambda(N-1)^{-1/N}]^2-\mathrm{i}2\lambda(N-1)^{-1/N}\sin\vartheta}{1-2\lambda(N-1)^{-1/N}\cos\vartheta+[\lambda(N-1)^{-1/N}]^2}\,\mathrm{d}\vartheta. \tag{60}$$

Exploiting periodicity to return to the limits to $\pm\pi$, one obtains

$$\dot{\lambda}/\lambda = \frac{1-[\lambda(N-1)^{-1/N}]^2}{2\pi\lambda B}\int_0^{\pi} \frac{1-2[\lambda(N-1)^{-1/N}]^N\cos N\vartheta+[\lambda(N-1)^{-1/N}]^{2N}}{(1+2\lambda^N\cos N\vartheta+\lambda^{2N})^{\frac{1}{2}}}$$

$$\times \frac{1}{1-2\lambda(N-1)^{-1/N}\cos\vartheta+[\lambda(N-1)^{-1/N}]^2}\,\mathrm{d}\vartheta. \tag{61}$$

As expected, this result is independent of n. It cannot be expressed in terms of convenient standard functions, but its numerical quadrature presents no problems, and a second quadrature gives the time.

For $N = 2$, (61) can be expressed in terms of an elliptic integral:

$$\dot{\lambda} = -\frac{1-\lambda^2}{2\pi B}\int_0^{\pi} \frac{1-2\lambda^2\cos 2\vartheta+\lambda^4}{(1+2\lambda^2\cos 2\vartheta+\lambda^4)^{\frac{1}{2}}}\frac{1}{1-2\lambda\cos\vartheta+\lambda^2}\,\mathrm{d}\vartheta \tag{62a}$$

$$= -\frac{1-\lambda^2}{2\pi B}\int_0^{\pi} \frac{(1+\lambda^2)+2\lambda\cos\vartheta}{[(1-\lambda^2)^2+4\lambda^2\cos^2\vartheta]^{\frac{1}{2}}}\,\mathrm{d}\vartheta. \tag{62b}$$

The $\cos\vartheta$ term vanishes, and after changing variables with $\cos\vartheta \to x$ and employing equations (3.152–3) and (9.126–3) of Gradshteyn & Rhyzhik (1980) one obtains

$$\dot{\lambda} = -\frac{1-\lambda^2}{\pi B}K\left(\frac{2\lambda}{1+\lambda^2}\right) = -\frac{1}{\pi}\lambda(1+\lambda^4)^{\frac{1}{2}}K(\lambda^2). \tag{63}$$

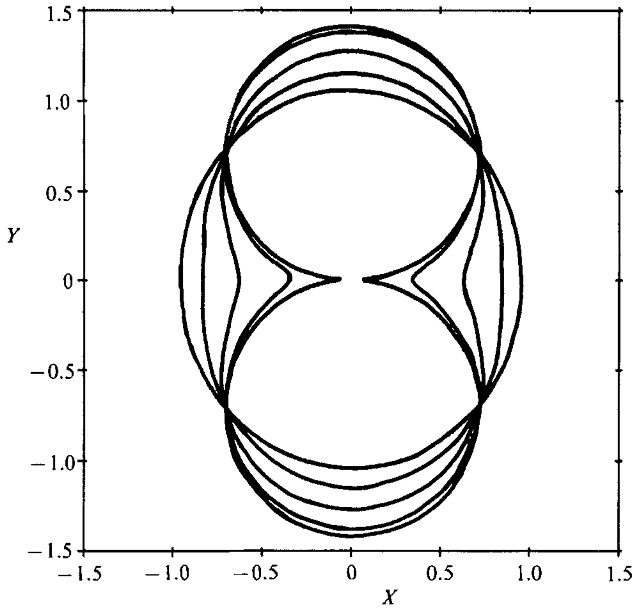

FIGURE 4. Coalescence of two cylinders (§4.4): $t = 0.04, 0.30, 0.80, 1.50, 3.00$.

Integrating, changing variables and replacing the parameter λ by $\nu = \lambda^2$, one has finally

$$t = \tfrac{1}{2}\pi \int_{\nu}^{1} [k(1+k^2)^{\frac{1}{2}} K(k)]^{-1}\,\mathrm{d}k, \tag{64}$$

where the shape is given by

$$z = \Omega[\sigma, \nu(t)] = \frac{1-\nu^2}{(1+\nu^2)^{\frac{1}{2}}} \frac{\sigma}{1+\nu\sigma^2}. \tag{65}$$

Equations (64) and (65) describe the coalescence of two equal cylinders in plane flow (figure 4) and are equivalent to results obtained previously by far more laborious means†. A generalization to unequal circles will be published (Hopper 1990*a*). These works provide additional figures and computational details, which are quite simple. Sills (1986) found that the predicted shapes for equal cylinders match experiment extremely well until about the point where the neck between the cylinders disappears. As his cylinders were free to shorten, it is plausible to attribute the subsequent failure to the plane-flow approximation becoming poor (cf. §5 and Hopper 1984). Sills did not accurately determine the time dependence.

Those who hold that Nature is benign will calmly accept all this; cynics will insist on the promised justification of the parameterization, equation (56). This is easily done. Using Lagrange's formula for the partial-fraction expansion, one finds $A_n = \lambda B/N$. Then (56) leads directly to

$$\frac{\dot{A}_n}{A_n} = -\frac{\dot{\lambda}}{\lambda}\frac{N\lambda^{2N}}{N-1}\left\{\left(1+\frac{\lambda^{2N}}{N-1}\right)^{-1} + \frac{2}{N-1}\left[1-\frac{\lambda^{2N}}{(N-1)^2}\right]^{-1}\right\}. \tag{66}$$

† Summarized in Hopper (1984). Even after re-writing the analysis in a clean form, it amounted to some thirty pages and will not be published. In that article, the parameter used is $\mu = \nu^2 = \lambda^4$, and the elliptic integral K is defined differently. It may be noted that in addition to the published errata, the expression after (7) in that work should read $\Lambda = \ln[16/(1-m^2)] \approx \ln[4\sqrt{2}/(x_n/R_0)]$.

Alternatively, (31) [using (56), $\alpha_n = \lambda(N-1)^{-1/N}\,\mathrm{e}^{-i\pi(2n-1)/N}$ and $\dot{\alpha}_n/\alpha_n = \dot{\lambda}/\lambda$], leads to the same result. This proves the contention. Direct substitution of (56) into (22) is a feasible alternative to using (30) and (31). In fact, the results of this subsection were originally obtained in that way.

4.5. *Polygons and an oval – functions multivalued on* $|\zeta| > 1$

Previously we considered the simplest classes of mapping functions, the polynomials and rational functions. As demonstrated by the general symmetric cubic, the solution of even these can present serious practical difficulties, but it was at least known that the parameterization itself was satisfactory. In this subsection we consider mappings whose solutions present more fundamental obstacles. It will be seen that the successful parameterization of many interesting and familiar maps evidently require more subtle ideas.

Both from the standpoint of applying the present formalism to practical problems and from that of understanding the fundamentals of this type of flow problem, functions multivalued on $|\zeta| > 1$ are of great interest, for they include regions with corners. More precisely, if the singular limit of $\Omega(\zeta, t)$, which we may denote $\Omega(\zeta, 0)$, is such that $z = \Omega(\sigma, 0)$ describes a curve $\Gamma(0)$ that has a corner (not a cusp) at the z-image of σ_0; then $\Omega'(\zeta, 0)$ will behave as $(\zeta-\sigma_0)^{\alpha}$, $-1 < \alpha < 1$, near σ_0. This does not mean that $\Omega(\zeta, t), t > 0$, must possess a singularity of this character anywhere. Nevertheless, as the singularities of polynomials and of the rationals considered above retain their character while moving in the ζ-plane, it is natural to attempt a parameterization along similar lines. Moreover, a pleasing qualitative appearance results.

An interesting family of maps is generated from

$$\omega'(\zeta) = (1-\zeta^N)^{\alpha} \quad (-1 \leqslant \alpha \leqslant 1), \tag{67}$$

together with the conditions $\omega(0) = 0$, $\lambda(0) = 1$ and (2). Then $z = \Omega(\sigma, 0)$ are sort of regular curvilinear polygons†. It turns out that only in the case of $\alpha = 1$ (the epicycloids already solved) is this parameterization correct. In all the other cases it fails, and for the same mathematical reason. The difficulty is conveniently displayed using the case $(\alpha = -1, N = 2)$.

Integration of (67) in this case gives

$$\omega(\zeta) = \tfrac{1}{2}\log\left(\frac{1+\zeta}{1-\zeta}\right) \quad (|\zeta| < 1), \tag{68}$$

where the principal branch of the logarithm is chosen, i.e. $\log 1 = 0$. With this choice, the description is unambiguous for $\lambda(t) < 1$. Setting the area at π, (24*a*) easily gives

$$B(\lambda) = \{\tfrac{1}{2}\ln[(1+\lambda^2)/(1-\lambda^2)]\}^{-\frac{1}{2}}. \tag{69}$$

† They have N-fold rotational symmetry and interior vertex angles $(1+\alpha)\pi$. If $\alpha = -2/N$, the Schwarz–Christoffel mapping from the unit circle to the interior of a regular polygon of N vertices and sides is obtained. The family $(\alpha = -1, N = 1)$, $0 < \lambda < 1$, are ovals having the same shapes as the $(\alpha = -1, N = 2)$ ovals, though at different values of λ and displaced to the right. In cases where $\omega(\zeta)$ cannot be expressed in terms of standard functions it is computationally effective to integrate numerically

$$\omega(\rho_0\sigma) = \sigma\int_0^{\rho_0} \omega'(\rho\sigma)\,\mathrm{d}\rho \quad (\rho_0 \leqslant 1)$$

at various values of ϑ.

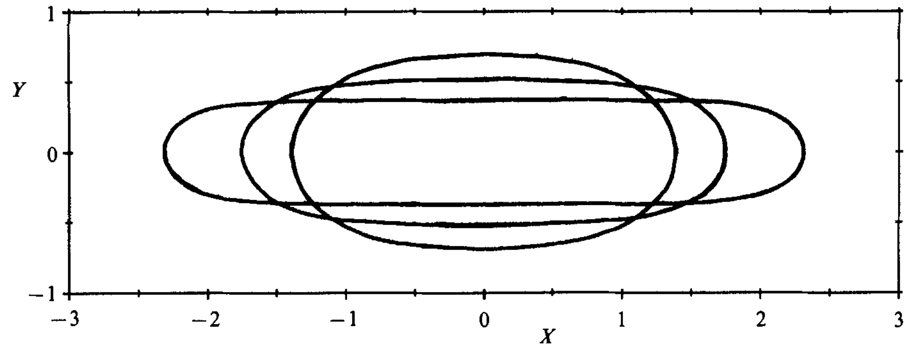

FIGURE 5. Symmetrical ovals – the $N = 2$, $\alpha = -1$ regular curvilinear polygon (§4.5). $\lambda = 0.9999$, 0.99, 0.9. (The ovals are more blunt and have flatter sides than ellipses of the same aspect ratio. As $\lambda \to 1$, the aspect ratio increases, the singular limit becoming the Re z-axis.)

These equations describe, for $\lambda < 1$, the family of ovals shown in figure 5. After carrying out the indicated operations, (22) leads almost directly to

$$\psi(\sigma,\lambda) = \left[\tfrac{1}{2}B(\lambda)\log\frac{\sigma+\lambda}{\sigma-\lambda}\right] \times\left\{\frac{1+\lambda^2\sigma^2}{1-\lambda^2\sigma^2}F(\sigma,\lambda)+\sigma F'(\sigma,\lambda)-\frac{\dot{\lambda}}{\lambda}\left[\frac{2\lambda B'(\lambda)}{B(\lambda)}+\frac{1+\lambda^2\sigma^2}{1-\lambda^2\sigma^2}\right]\right\} -\frac{\lambda B(\lambda)\,\sigma}{\sigma^2-\lambda^2}\left[F(\sigma,\lambda)+\frac{\dot{\lambda}}{\lambda}\right]. \tag{70}$$

The log function still refers to the principal branch. The analytic continuation of (70) onto $|\zeta| < 1$ gives $\psi(\zeta,\lambda)$ as required by the conjectured parameterization. The last term leads to simple poles at $\zeta = \pm\lambda$, which could be avoided by taking $\dot{\lambda}/\lambda = -F(\lambda,\lambda)$. This is inadequate, however, for the first factor of (70) means that $\zeta = \pm\lambda$ are also logarithmic singularities. That these are infinities is a relatively minor point; the real problem is that they are also branch points. As a result, the analytic continuation of $\psi(\sigma,\lambda)$ as given by (70) is multivalued on $|\zeta| < 1$. The only way the logarithmic factor would not lead to $\psi(\sigma,\lambda)$ being multivalued would be if the factor in braces were identically zero, and it is easily verified that it is not. As noted previously, $\psi(\zeta,t)$ being multivalued on $|\zeta| \leqslant 1$ implies unacceptable discontinuities in the physical state of the body. It follows that the conjectured parameterization does not describe a shape evolution arising from the type of flow under consideration.

One may also think of the difficulty as being a discontinuity across a particular branch cut, such as ($|\mathrm{Re}\,\zeta| < \lambda$, $\mathrm{Im}\,\zeta = 0$). Then $\psi(\zeta,\lambda)$ could be acceptable only if the factor in braces were vanish all along the cut. This cannot happen: if the factor in braces vanished on the cut, then analytic continuation of that factor off the cut would be identically zero, including on the unit circle.

Exactly analogous things happen with mappings having corners, including regular curvilinear polygons with $-1 < \alpha < 1$. One might hope that one could obtain a cancellation in the case of certain algebraic maps (e.g. $\alpha = -\frac{1}{2}$), but this is not the case. More generally, if $\Omega(\zeta,t)$ has branch points on $|\zeta| > 1$, then the analytic continuation of $\Omega(\sigma,t)^*$ onto $|\zeta| < 1$ leads to corresponding branch points in the latter region. And branch points on $|\zeta| < 1$ lead to intractable incompatibilities of the shape-evolution criteria. Indeed, referring to (22), it would seem extremely difficult

– and perhaps mathematically impossible – to construct $\dot{\Omega}(\zeta, t)$ such that $\psi(\zeta, \lambda)$ meets the requirement of being single-valued.

Though the foregoing is by no means a proof, it appears that the shapes for $t > 0$ are such that $\Omega(\zeta, t)$ can have no branch points on $|\zeta| > 1$. This means that the mapping function $\Omega(\zeta, t)$ describing the shape at any observable time (that is, after any finite amount of flow) must be single-valued everywhere in the ζ-plane. This conclusion is of fundamental interest and serves also as a useful guide in guessing maps. The reader will rightly treat this conjecture with caution, but the argument is compelling.

4.6. *Entire and meromorphic mapping functions*

Unlike the multi-valued functions, there is little practical incentive for studying these, but they are of theoretical interest. By definition, an entire (or 'integral') function is one that is analytic in the entire complex plane (except for the point at infinity), and a meromorphic function is one whose only singularities in the entire complex plane (except for the point at infinity) are poles. (See e.g. Titchmarsh 1939, chapter 8.) Thus, the simplest entire functions are the polynomials, and the simplest meromorphic ones are the rational functions. Since polynomial and rational maps evolve with the form unchanged, in the sense of §3, one might hope for something similar. This does not appear to be the case, as illustrated by the two elementary examples below. The problems, as might be expected, are associated with the facts that a non-polynomial entire function has an essential singularity at infinity, while a non-rational meromorphic function has a limit of poles there.

The function $\sin \lambda\zeta$ is entire (and of order 1), and the mapping $z = B(\lambda) \sin \lambda\sigma$, which resembles the nephroid, is simple for $\lambda < \frac{1}{2}\pi$. However, the analytic continuation of $\Omega(\sigma, \lambda)^* = B(\lambda) \sin(\lambda/\sigma)$ has an essential singularity at $\zeta = 0$. Similar problems arise from the other conjugate quantities appearing in (22). An elementary meromorphic function is $\tan \lambda\zeta$, which has simple poles at $z = (2n+1)\pi/2\lambda$. The mapping $z = B(\lambda) \tan \lambda\sigma$ is simple for $\lambda < \frac{1}{2}\pi$. In this case, the analytic continuation of $\Omega(\sigma, \lambda)^*$ leads to an (enumerable) infinity of poles on $|\zeta| < 1$ (on the Re-ζ axis, and accumulating at the origin). In both cases, the resulting singularities cannot be removed under this rather naive parameterization. Moreover, the generality of the difficulty (the essential singularity of a non-polynomial entire function at infinity, and the limit of poles of a non-rational meromorphic function there) raises the question of whether a suitable parameterization of any non-polynomial entire function or of any non-rational meromorphic function is possible.

5. Discussion

When successful, the methods presented here give exact solutions to the mathematical problem stated at (5), but this problem involves a number of physical approximations worth noting. First, the condition that $\Omega(0, t) = 0$, introduced for convenience, in some cases leads to rigid-body accelerations. This is an artifact: it has no effect on the shape evolution and can always be removed *ad hoc* at the end of an analysis. Secondly, the Suratman number can often be made small independent of size R_0 by raising the viscosity η (for example, by a temperature change); a small Bond number can only be achieved with small size or a microgravity environment. One ought to verify that the inertial and gravitational terms of (4) remain small for a calculated flow. Thirdly, the surface tension of a real liquid depends significantly on the local curvature if it is large enough. Also, two surfaces of a real liquid in close proximity exert a force on each other; that is, their free energy per unit area depends on the separation of the surfaces, and so, therefore, does the surface tension. These phenomena arise from London–van der Waals' forces, surface dipole layers, adsorbed molecules, etc. and more generally from the fact that the environment of a material volume element at a surface is different when the surface is flat and isolated from when it is not. Unfortunately, it is the regime of small sizes, where inertial and gravitational effects become negligible, that the constancy of the surface tension is most likely to come into question. Fourthly, the liquid must be Newtonian and incompressible over the regime of stresses encountered. Typically, stresses high enough to be a concern would be generated only by curvatures so large that the surface tension would not be constant. Finally, there is the assumption of plane flow. The plane-flow approximation is usually invoked for long cylinders constrained against flowing in the axial direction – by inertia in the case of 'infinitely' long cylinders. If the ends of the cylinder are free, then capillarity induces, in addition to incidental flows near the ends, a general axial flow shortening the body and increasing its cross-sectional area. In assessing the applicability of the plane-flow approximation, one should estimate this axial flow and compare its velocities to the in-plane velocities of the theory.

In the coalescence of cylinders, for example, one might expect the following: (i) the in-plane flows due to the axial flow would be negligible compared with those calculated in the plane-flow approximation at small times, but that the opposite would be true when the shape was nearly circular ($t \to \infty$); (ii) the range of the parameter ν over which the constancy of surface tension is a good approximation will depend on the size R_0 in ways that depend on the liquid itself; and (iii) if the cylinders are small enough for gravitation effects to be small, then inertial and non-Newtonian effects will be negligible.

The class of problem analysed is rather restricted: the dynamics (plane creeping flow driven by surface tension alone) is relatively simple; and the regions are the simplest. However, the class is also very intriguing for its self-contained nature – that the tractions arise from the geometry. One cannot reasonably expect to extend the methods of the present work to other dynamics, but it would be very desirable to extend the results in other ways. Extension to infinite domains is straightforward (Hopper 1990*a*). General three-dimensional flows would require a generalization of the concepts of conformal mapping and analyticity to higher spatial dimensions. This topic has not escaped the attention of mathematicians, but it is not clear that any of the available generalizations would lead to simple solutions of the creeping capillary flow problem. Axisymmetrical flows, involving only two spatial coordinates, can be described in terms of ordinary conformal maps, and the use of Stokes' stream function (Lamb 1932 §94; Garabedian 1966 p. 426) seems natural. Extensional flows are another possibility, a special case of which is when the general cylinder is allowed to shorten under the influence of surface tension at its ends. Extension of the present formalism to plane flow in doubly-connected regions is straightforward, for there is always a mapping from an annulus. The author has not examined any of these ideas in detail. Reflecting on the intrinsic connection between the geometry and dynamics of this class of problems, it seems important to determine more precisely the constraints on $\Omega(\zeta, t)$. To speculate, it may turn out that $\Omega(\zeta, t)$ must be a rational function with no repeated factors in either the numerator or denominator. While it has seemed to the author natural to focus on the trajectories of critical points, this has so far been unenlightening. The trajectories of critical points in a certain mapping from $\operatorname{Im} \zeta \geqslant 0$ are described in Hopper 1990*a*.

A different approach involves the idea of finding variational principles. Intuitively, one feels there must be something special here. After all, it is reasonable to expect a region flowing under the influence of surface tension to reduce its total surface energy as steeply (in some sense, perhaps that of time) as possible; while by Helmholtz's principle, the flow must be such as to dissipate this energy (as heat) as slowly as possible.† It is intriguing that our plane flows naturally become circular, while the inverse mapping $\zeta = \Omega^{-1}(z)$ is governed by either of two variational principles involving the circle (Carrier *et al.* 1966 §§4–8). These involve extremizing integrals of $|\mathrm{d}\Omega^{-1}(z)/\mathrm{d}z|$ around the contour Γ or of $|\mathrm{d}\Omega^{-1}(z)/\mathrm{d}z|^2$ over the region D. Does the function F, involving $|\Omega'(\zeta)|^{-1}$, have some interesting interpretation? Recall also the easily proved facts that, if the (dimensionless) perimeter of the region D is L, then the (dimensionless) pressure averaged over the perimeter is $L/2\pi$, while that averaged over the area is $2\pi/L$. These are equal only in the static circular case, when $L = 2\pi$. It might also be interesting to consider variational principles applied to the three-dimensional tube $[X(t), Y(t), t]$, $0 < t < \infty$. Is it, perhaps, some sort of constrained minimum?

The theory and methods presented in this article are intended to provide exact solutions of this class of fluid dynamical problem. Two types of approximations can be introduced. The first is exemplified by the symmetrical general cubic: one has in principle simple equations describing the flow exactly, but it is found necessary to solve them numerically. This is a problem in ordinary numerical analysis. The second type of approximation occurs when, as in the §4.5, there is no obvious parameterized map for the given initial shape. One then seeks a mapping function giving an acceptable approximation to the shape of interest, and then determines the evolution of that mapping using exact formulae (possibly solved with a numerical analysis of the first type approximation). This is a problem in the general field of approximate conformal maps, any detailed consideration of which would take us far afield from our subject matter (cf. Trefethen 1986). A few comments are, however, appropriate. One expects the flows from different maps approximating the same starting shape to converge with time. That is, suppose we have two maps $\Omega_1(\zeta, t)$ and $\Omega_2(\zeta, t)$, both of which approach (in some sense) the same (possibly singular) shape as $t \to t_1$. Then one expects that $\Omega_1(\zeta, t) \to \Omega_2(\zeta, t)$ (again, in some appropriate sense) as $t \to \infty$. Approximations giving ζ in terms of a polynomial in z seem unnatural in the present context, for then the mapping function giving z in terms of ζ will be multi-valued. Truncated Taylor series in ζ are the simplest approach, but many terms are needed to approximate closely singular shapes of interest. It is also natural to consider approximating maps with rational functions (Rudin 1966), because the latter were shown to evolve in a simple manner, but Padè approximants, at least, seem not much better than truncated Taylor series. In any case the practical difficulties in obtaining detailed time dependence remains. Whether combinations of the present concepts with modern map approximation methods will prove fruitful remains to be seen.

An interesting point, with implications to the theory of glass strength, is whether anything can be said about a simply-connected region remaining so. A hole cannot generate continuously, for it is not possible to map between domains of different connectivities. Also, on the basis of the model physics incorporated into the evolution equation, it cannot be possible for a cusp to form continuously from a simple curve in the forward time direction: tractions and stresses at the surface would blow up.

† Helmholtz's principle is, in the present context, equivalent to the thermodynamic principle of minimum entropy production and to the elasticity principles of minimum strain energy, minimum potential energy, and minimum complementary energy (see Lamb 1932, §344; Sokolnikov 1956 chap. 7).

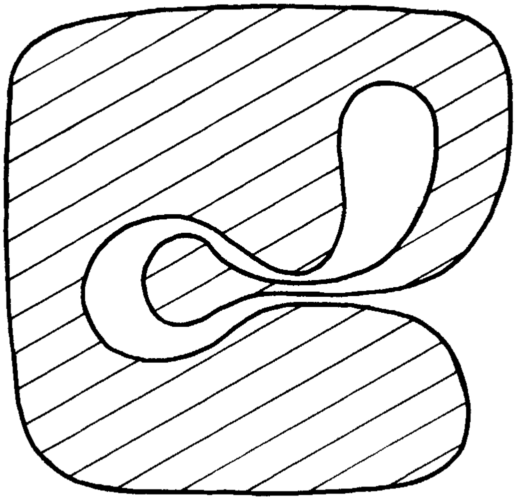

FIGURE 6. Example of a possible transition from a simple to a non-simple shape (§5): Does the globe extract itself from the mouth without hitting the walls?

Tangential contact, however, is another matter. Here the idea is that distinct portions of the curves could contact, forming a pair of cusps discontinuously, at which point the mathematical theory would become inapplicable to the model physics. In an admittedly crude experimental analogue, I have failed to observe such an occurrence in the (slow!) flow of oil floating on water. A worrisome configuration is depicted in figure 6. Mathematically, this may be a deep and difficult question. Global and not merely local properties of the maps are involved. There are quite a few famous theorems giving interesting properties of univalent functions, but the converse is not true: There are very few theorems whose hypotheses imply a simple mapping (cf. Henrici 1986, chapter 19). Having made no significant progress with this problem, I merely offer the following conjecture on simplicity: By hypothesis, the initial map $z = \Omega(\sigma, t_1)$ is simple; then the shape-evolution condition (equations (21) or (22)) implies that the map $z = \Omega(\sigma, t), t > t_1$, is also simple. A weaker conjecture would require that $z = \Omega(\sigma, t_1)$ be a star domain, in which case the intuitive appeal is stronger.

6. Summary

The creeping viscous incompressible plane flow in a finite region, bounded initially by a simple smooth closed curve and driven solely by surface tension was analysed. The shape evolution of the boundary is described by a time-dependent conformal mapping $z = \Omega(\zeta, t)$. An equation (equivalently equations (21) or (22)) determining the time evolution of $\Omega(\zeta, t)$ was derived. It was argued that (i) a shape given initially by an Nth-order polynomial map will be described by an Nth-order polynomial map at all subsequent times; (ii) a shape given initially by an N-term partial-fraction expansion will remain such at all subsequent times; (iii) a shape given initially by a mapping that is the ratio of an Mth- to an Nth-order polynomial, with $M \leqslant N$ and having no common or repeated factors, will be described by this same form *at all* subsequent times; and (iv) the map $\Omega(\zeta, t)$ describing the shape at any observable time must be single-valued everywhere. Indeed, it seems likely that the map must be a rational function. Solved examples were given, including the coalescence of equal cylinders. It is conjectured that a region simply-connected initially must remain so.

The author thanks Professor S. Richardson (University of Edinburgh) and an anonymous reviewer for helpful comments on an earlier manuscript. Work performed under the auspices of the US Department of Energy by the Lawrence Livermore National Laboratory under contract W-7405-ENG-48.

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